United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 29 November 2013

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers, set squares and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top* left *hand corner.*
- *To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 8am GMT on Saturday 30 November.*

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2013/14 British Mathematical Olympiad

Round 1: Friday, 29 November 2013

1. Calculate the value of
$$\frac{2014^4 + 4 \times 2013^4}{2013^2 + 4027^2} - \frac{2012^4 + 4 \times 2013^4}{2013^2 + 4025^2}.$$

2. In the acute-angled triangle ABC, the foot of the perpendicular from B to CA is E. Let l be the tangent to the circle ABC at B. The foot of the perpendicular from C to l is F. Prove that EF is parallel to AB.

3. A number written in base 10 is a string of 3^{2013} digit 3s. No other digit appears. Find the highest power of 3 which divides this number.

4. Isaac is planning a nine-day holiday. Every day he will go surfing, or water skiing, or he will rest. On any given day he does just one of these three things. He never does different water-sports on consecutive days. How many schedules are possible for the holiday?

5. Let ABC be an equilateral triangle, and P be a point inside this triangle. Let D, E and F be the feet of the perpendiculars from P to the sides BC, CA and AB respectively. Prove that

   a) $AF + BD + CE = AE + BF + CD$ and
   b) $[APF] + [BPD] + [CPE] = [APE] + [BPF] + [CPD]$.

   The area of triangle XYZ is denoted $[XYZ]$.

6. The angles A, B and C of a triangle are measured in degrees, and the lengths of the opposite sides are a, b and c respectively. Prove that
$$60 \le \frac{aA + bB + cC}{a + b + c} < 90.$$